CAS NL #35, May 1981

THE ACOUSTICS OF STRINGED INSTRUMENTS STUDIED BY STRING RESONANCES

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Introduction

It is well known that vibrating strings on musical instruments radiate very little direct sound but rely instead on the transmission of energy to the acoustically radiating surfaces of the instrument as a whole, via the bridge, to provide appreciable acoustic radiation. The strength of this coupling and its variation with frequency are important factors in determining the tonal quality of any instrument. Consequently, by studying the way that this coupling affects the resonance frequency and damping of the vibrating strings, a great deal of useful information can be learned about the properties of an instrument that determine its tone.

The wolfnote is undoubtedly the most dramatic illustration of the way that this coupling can affect the tone of an instrument of the violin family. It was Schelleng's classic analysis of the wolfnote phenomenon[1], in terms of the resonant response of a string over-strongly coupled to a structural resonance, which stimulated our initial interest in the possibility of using string resonances to derive information about the acoustics of a violin (see Benade[2] for an up-to-date account of our present understanding of the wolfnote phenomenon).

Hancock[3,4] appears to have been the first to appreciate the potential of string resonance measurements as a method for obtaining quantitative information about the acoustically important resonances of a stringed instrument. Using an elegant technique, based on the Doppler-shift of a laser beam, Hancock[3] was able to demonstrate that string resonances, in practice, are rather more complicated than might have been expected from earlier text book treatments (as, for example, given by Rayleigh[5]).

In this paper we describe a rather simple experimental technique that can be used to obtain highly sensitive and reproducible measurements of string resonances on any instrument of the violin family or on related instruments. We also describe a simple model for interpreting such measurements, allowing us to derive many of the important properties of the structural resonances of an instrument that determine its tone.

Experimental technique.

As experimental details have been given elsewhere[6,7], we simply show Fig. 1 to illustrate the general experimental arrangement and confine our comments to the more important experimental features.

A simple photo-detector system is used to monitor the string vibrations, which are excited electromagnetically by passing an alternating current along the metal, or metal-covered, string, with a magnet providing a localized Lorentz force at a position near the bowing point. It is possible to vary the orientation of the magnet to give an exciting force in any chosen direction in the plane perpendicular to the string.

The electronic detection system allows us to monitor the components of string velocity, v, in-phase and at 90^{o}-phase with the exciting force, F. If the velocity is measured at the same point as the string is excited, we obtain the effective complex mechanical admittance, $v/F = A' + jA''$, at the point of string excitation. The two components of string velocity, giving A' and A'', are plotted continously on an xy-recorder as the frequency, $f = \frac{\omega}{2\pi}$, of string excitation is swept slowly through the resonance. A typical resonance curve is generally scanned in about 30 seconds to avoid problems from ringing.

All our measurements are made on the total string length between bridge and nut. The string resonances are varied in frequency over any chosen range of interest by varying the tension and by the use of the natural string harmonics. Measurements on a number of violins and cellos have been made over a frequency range from 100 Hz to 4kHz.

The use of the full string length avoids the problems that we initially experienced when making measurements on artificially-stopped lengths of string. In practice, we found that such measurements were never very reproducible, since it proved difficult to produce an adjustable end-stop with reproducible terminating characteristics. Fortunately, the nut on a violin or cello appears to be almost ideally designed to provide a perfectly rigid end-stop for both transverse and torsional vibrations of the string. This is certainly not true for the finger-fret combination on the guitar and other fretted instruments, where the ability of the player

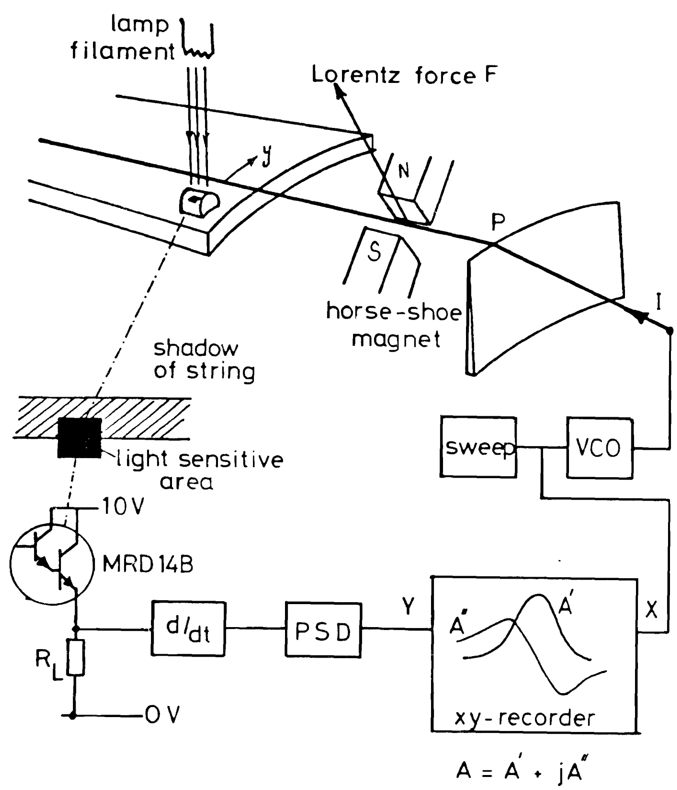

FIG.1: Experimental arrangement

to vary the terminating impedance is almost certainly an important factor in enabling the player to produce a vibrato effect, even though the player's finger lies behind the fret[8].

Although it might be thought that varying the tension of the strings on an instrument would change the structural resonances that the measurements are designed to study, this is not a significant effect in practice, for reasons that will be explained later in this paper.

Measurements.

Fig. 2 shows some typical resonances for a Pirastro G-string mounted on a violin with a relatively strongly coupled body resonance at about 465 Hz. The measurements were made using the n=2 mode of string vibration, where the unperturbed resonant frequency is given by $f_n = nc/2l$ (n is the order of the mode, c is the velocity of transverse waves on the string, and l is its length).

These measurements immediately suggest the superposition of two quite distinct resonances, which, for reasons that we shall explain, we choose to refer to as the "coupled" and "uncoupled" modes of string vibration. Such resonances have a straightforward explanation in terms of the coupling between the string and structural resonances, as we shall now explain.

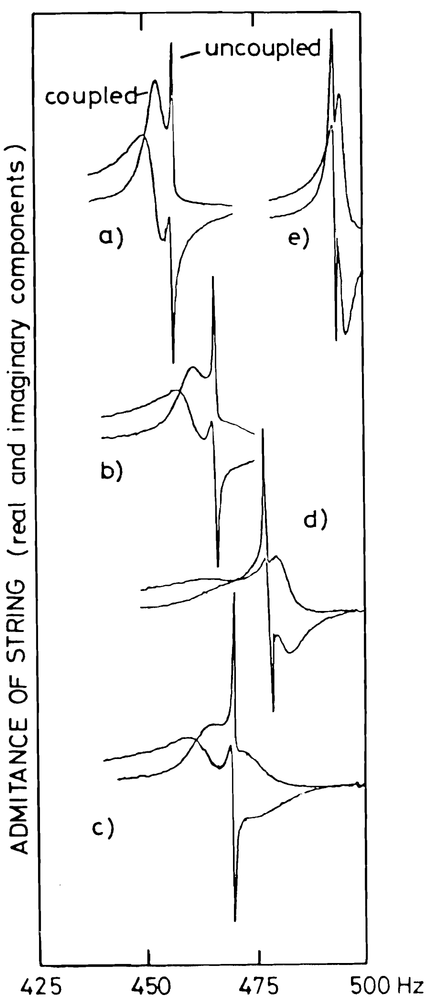

FIG.2: Sequence of n=2, violin G-string resonances in vicinity of main body resonance

We first note that at low frequencies the resonances of an instrument that are most likely to be strongly coupled are the air and main body resonances, both of which involve the motion of the bridge illustrated schematically in Fig. 3. The vibrations of the body of the instrument cause the bridge to rock in its own plane about a position close to the foot of the bridge nearest the soundpost. Such motion results in P, the point of string support on the bridge, moving in a direction at an angle Θ_B to the normal bowing direction. Structural resonances at higher frequencies will involve motion in other coupling directions, which will depend on the vibrational characteristics of the bridge itself at higher frequencies (Reinicke[9]). As we are only interested here in low amplitude linear response, we ignore any induced motion of P parallel to the length of the string, as this will only produce second-order corrections to the frequency and damping of the string resonances.

If we consider coupling to a single structural resonance alone, it is easy to explain the two superimposed string resonances observed in the measurements of Fig. 2. For string vibrations in a direction perpendicular to the coupling or rocking direction, the point of string support P remains a perfect node, so that the resonances for vibrations in this direction remain sharp and unperturbed in frequency. However, for string resonances in the coupling direction, the bridge moves so that P is no longer a node, resulting in a shift and broadening of the resonance.

In Fig. 2a we identify the broader resonance at the lower frequency as the "coupled" string resonance and the sharper resonance as the "uncoupled" resonance. If a string with these characteristics is plucked, both modes will, in general, be excited leading to an initial decay of sound associated with the "coupled" resonance superimposed on a more slowly decaying signal from the uncoupled mode. Although the uncoupled mode will not be coupled to the principal structural resonance ex-

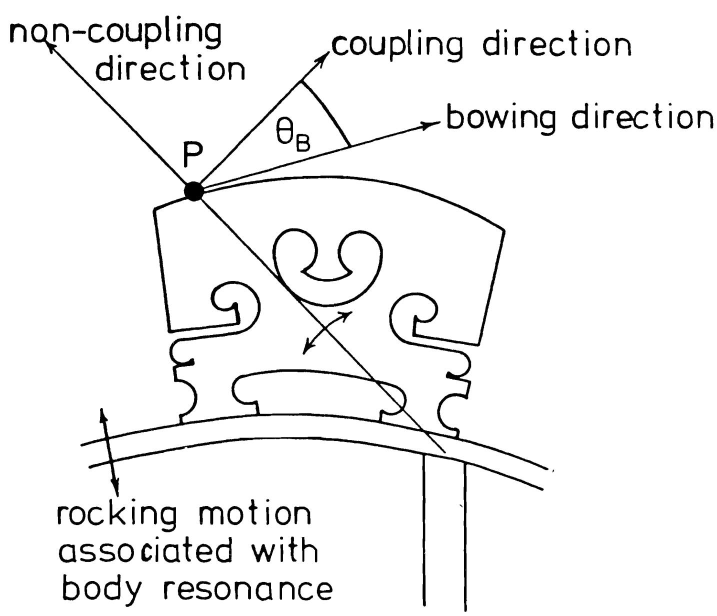

FIG.3: Coupling induced by string vibrations near main body resonance

cited, it will, nevertheless, be weakly coupled to other structural resonances involving other coupling directions and will thus radiate sound. The double decay of sound from a plucked violin string is evident in Reinicke's measurements[9]. Related effects have also recently been reported for the piano by Weinreich[10], but in this case the situation is further complicated by the interactions between almost identically tuned strings.

In general, the string will be coupled, if only weakly, to a number of structural resonances each involving a particular coupling direction. It is straightforward to show that the string continues to support two independent modes of transverse string vibration, which are very nearly linearly polarized in orthogonal directions[8]. Whenever a string resonance lies close to a coupled structural resonance, the two modes of string vibration will be polarized in directions close to the coupling and non-coupling directions of the structural mode principally excited. Thus by varying the direction of the exciting force, so that only one of the two possible modes are excited, it is possible to determine the coupling direction relative to the bowing direction, which is the important angle in practice. The perturbation in frequency and width of the coupled string resonances then provides all the information necessary to evaluate the acoustically important parameters of the structural resonance excited, as we shall now describe.

Derivation of acoustic parameters.

To relate the perturbed frequencies and widths of string resonances to the properties of the coupled structural resonances, we consider the transmission line analogue used by Schelleng[1], shown in its simplest form in Fig. 4a. We assume coupling to a single structural resonance and ignore coupling to the air resonance, though it is straightforward to extend the model to include such coupling. We also confine our discussion to string vibrations in the coupling direction. The string is considered as a transmission line of characteristic impedance Z_0 = cm/l, where m is the mass of the vibrating string. The string is terminated at the nut-end by an open-circuit and at the bridge by a series resonant circuit representing the impedance at the point of string support of the structural resonance, where M_B, C_B and R_B are the effective mass, compliance and resistance respectively. The vibrational characteristics of the bridge are considered to be included as contributions to the structural resonance.

The terminating impedance can usefully be represented as an equivalent length ε of resistively terminated transmission line, as indicated in Fig. 4b. Close to the structural resonance the circuits shown in Fig. 4a and 4b are equivalent with the parameters given by

$$\frac{\varepsilon}{\ell} = \frac{1}{n\pi} \cdot \frac{Z_B}{R_B} \cdot \frac{2\, Q_B\, \delta}{\left[1 + (2\, Q_B\, \delta)^2\right]} \qquad (1)$$

and

$$r = R_B \left[1 + (2\, Q_B\, \delta)^2\right] \qquad (2)$$

where $\delta \approx (f_B - f)/f_B$, and $Q_B = \omega M_B/R_B$, and all correction terms of order Z_0/R_B have been dropped, as is always justified for any stringed instrument.

The interpretation of the above equations is straightforward. For string resonances at a lower frequency than the coupled structural resonance, the bridge at the point of string support P is forced to move in-phase with the force exerted on it by the vibrating string, as shown schematically in Fig. 5a. The effective string length between nodes is therefore increased by an amount ε given by eq. (1). This explains why the "coupled" resonance in Fig. 2a is at a lower frequency than the "uncoupled" resonance. However, when the string resonance is at a higher frequency than the structural resonance, the bridge moves in anti-phase with the string, as illustrated in Fig. 2b. The effective resonating string length is thus shortened, ε<0, giving a resonance at a higher frequency than the "uncoupled" string resonance. The sequence of string resonances in Fig. 2 illustrates the way that the relative positions of the "coupled" and "uncoupled" resonances reverse as one passes across the coupled main-body resonance at ≈ 465Hz.

4(a)
4(b)

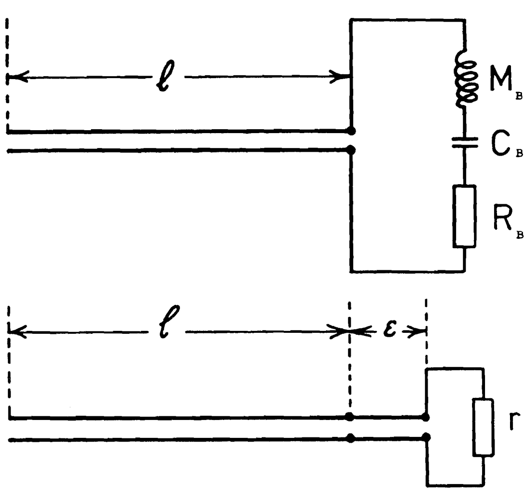

FIG.4(a): Transmission line analogue of string coupled to a structural resonance

FIG.4(b): An equivalent representation of the above circuit

5(a)
5(b)

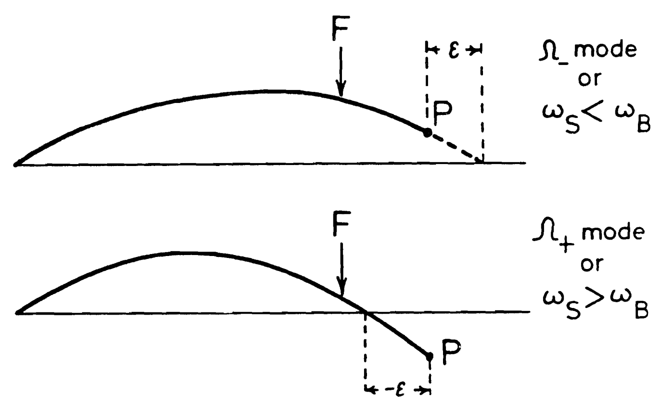

Fig.5a & 5b:

Motion of string and point P of string support on the bridge for frequencies a) below and b) above the frequency of the coupled structural resonance.

To proceed further it is helpful to distinguish between string resonances that are weakly or strongly coupled to a structural resonance.

(i) Weak coupling.

For weak coupling, we assume that any additional damping from the coupling is small, so that string resonances will typically have Q-values well in excess of 100. Over the width of the string resonance the effective impedance presented by the structural resonance, with typical Q-values of 20-50, can therefore be considered constant. Under these conditions the resonances can be considered as those of a string of effective length $l+\varepsilon$ terminated by a purely resistive impedance r, where ε and r are given by the above equations.

To a good approximation, the perturbation in frequency of the "coupled" string resonance, $\Delta f/f = -\varepsilon/l$, can be determined from the separation in frequencies of the "uncoupled" and "coupled" resonances.

Eq. (2) shows that the damping of the "coupled" string resonance will be largest when the frequencies of the string and structural resonances coincide. This is illustrated experimentally by the sequence of weakly coupled string resonances shown in Fig. 6. These measurements were obtained with the magnet aligned to excite the "coupled" string resonance only. The string was a Pirastro E-string slackened until its n=2 mode of vibration was close to a weakly coupled structural resonance of the instrument at $\approx$ 634Hz - the frequency at which the damping of the string resonances is seen to be largest. In the vicinity of a coupled resonance, the width of the string resonances at half weight, Γ, can be expressed as

$$\Gamma = \frac{\Gamma_B}{[1+(2\delta Q_B)^2]} + \Gamma_o + a\delta , \qquad (3)$$

where

$$\Gamma_B = \frac{2 f_B}{n\pi}\cdot\frac{Z_o}{r} = \frac{f_B}{(n\pi)^2}\cdot\frac{m Q_B}{M_B} \qquad (4)$$

The first term in eq. (3) gives the damping from the resonant coupled modes, while the second and third terms represent any background damping effects including the damping from non-resonant structural modes. By fitting the widths (or inverse heights) of the string resonances to expressions of the above form, it is straightforward to derive values for f_B, Γ_B and Q_B, and from the known mass of the string to derive M_B. From the resonances shown in Fig. 6 we obtain the following values for this particular structural resonance; f_B= 634Hz, $Q_B \approx 30$ and $M_B \approx$ 125g.
From the angle of the magnet required to excite the "coupled" mode of string resonance alone, we obtain a coupling direction relative to the normal bowing-direction of $\approx 45^o$. Similar information may be obtained for at least the first half-dozen or so of the weakly coupled resonances for instruments of the violin family.

(ii) Strong coupling.

When the coupling is sufficiently strong, the damping of string resonances is so large that the terminating impedance can no longer

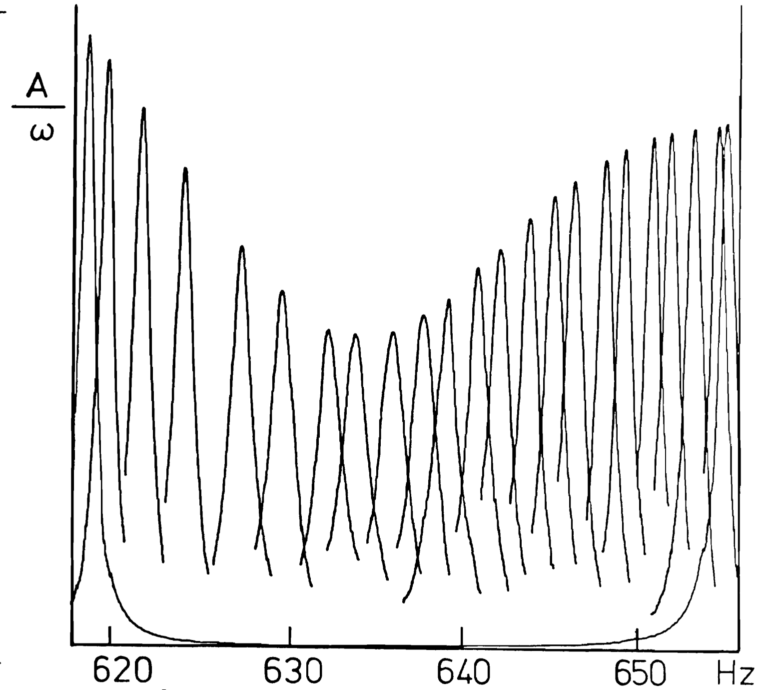

FIG.6:

A sequence of violin string resonances (real part of A) in the vicinity of a weakly coupled structural resonance at 634 Hz

be considered constant over the width of the string resonance. If we consider a string with unperturbed resonant frequency coincident with that of a strongly coupled structural resonance, $f_S = f_B$, eq.(1) shows that, as the excitation frequency f moves away from f_B, the effective string length, $l + \varepsilon$, moves to give a perturbed resonant frequency moving away from f_B in the same direction as f is changed. For sufficiently large coupling, the perturbed resonant frequency moves away from f_B even faster than f, so that two new positions of resonance are eventually approached above and below f_B, where the damping of string vibration, given by eq. (2) is smaller than at the center frequency f_B. This accounts for the double string resonance predicted by Schelleng[1] using a numerical evaluation of the transmission line problem. Schelleng pointed out that this feature is simply the familiar double resonance of a coupled, but relatively lightly damped, pair of mechanical oscillators. Much earlier, White[11] and Raman[12] had discussed the existence of the wolfnote in terms of the properties of such a system.

Fig. 7 shows an example of a double resonance for the n=2 mode of a cello C-string on an instrument with a strongly coupled structural resonance known to be responsible for a troublesome wolfnote. Although the direction of the magnet has been adjusted carefully, the "uncoupled" mode of string resonance has not been suppressed completely and appears as a very much sharper resonance between the split resonances of the "coupled" string vibration. Similar measurements on a violin with a wolfnote have been successfully interpreted with a numerical evaluation of the transmission line problem[7]. We prefer here to consider such measurements in terms of the predictions of an analytic model for the splitting and damping of the normal modes of the coupled system, which will be presented in more detail elsewhere[8].

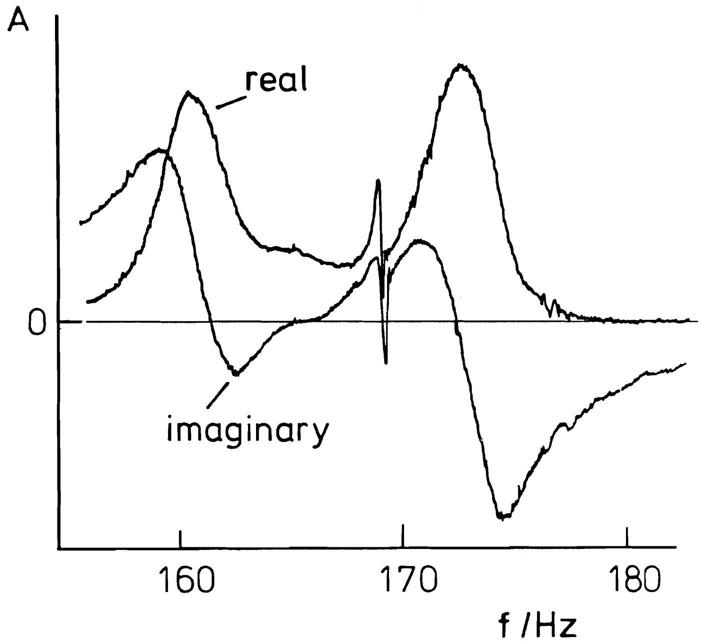

FIG.7: n=2 resonance of a cello C-string in neighbourhood of a strongly coupled main body resonance

Again for simplicity, we initially consider the response of the coupled system, when the unperturbed string resonance and coupled structural resonance coincide, and we ignore any contribution to the damping other than that associated with the coupled oscillators. The frequencies $f_{\pm}$ of the damped normal modes of the system are then given by the following simple expression

$$\frac{f_{\pm}}{f_B} = \frac{\Omega_{\pm}}{\omega_B} = \left[1 + \frac{1}{2\,Q_B}\left(j \pm \sqrt{K^2 - 1}\right)\right]^{1/2} \qquad (5)$$

where K is equivalent to the coupling factor used by Meamari[13] and Guth[14] in an interesting theoretical analysis of normal mode splitting and wolfnote production on a specially prepared laboratory monochord. For our purposes it is convenient to express K as

$$K = \frac{2\,Q_B}{n\pi}\sqrt{\frac{m}{M_B}} \qquad (6)$$

Eq. (5) indicates that the character of the normal modes depends on the magnitude of K relative to unity. For weakly coupled systems ($K \ll 1$), the normal modes can be identified with the structural response at f_B, with Q-value $\approx Q_B$, and the string resonance at the same frequency, with a Q-value

$$Q_S = \frac{4\,Q_B}{K^2} = \frac{(n\pi)^2}{Q_B} \cdot \frac{M_B}{m}, \qquad (7)$$

which is equivalent to the result obtained in the previous section, eq. (4).

When K>1, it is no longer possible to consider the resonances of the coupled system as distinguishable string and structural resonances. The two modes are now equally damped with Q-values = $2Q_B$ but with resonant frequencies now split symmetrically about f_B with a spacing approaching

$$\frac{f_+ - f_-}{f_B} = \frac{K}{2\,Q_B} = \frac{1}{n\pi} \cdot \sqrt{\frac{m}{M_B}} \qquad (8)$$

in the limit of large K. The vibrations of the system corresponding to these two modes can be schematically represented by Figs. 5a and 5b, where, for the Ω_- phase the bridge and string move in phase with each other, while for the Ω_+ mode they move in anti-phase.

From the separation in the resonant frequencies of the two modes and their damping, it is straightforward to derive values for the frequency of the strongly coupled structural resonance, f_B, its Q-value, Q_B, and its effective mass, M_B. For the measurements shown in Fig. 7 we obtained f_B=166Hz, $Q_B \approx 23$ and $M_B \approx 94$g.

Admittance curves and the bowed string.

A particularly interesting feature of the measurements shown in Fig. 7 is the way that the imaginary component of the admittance for the "coupled" double resonance passes through zero three times. Schelleng[1] suggested that this behaviour was intimately connected with the occurrence of the wolfnote when such a string was bowed. The minimum value of K to produce such a feature, K=2, is twice the value required to give a split resonance. However, the value of K required to give the triple zero-crossing feature in practice also depends on the position of excitation, as may easily be understood by reference to Figs. 5a and 5b. When the string is excited close to the bridge, as when the string is bowed, the string is excited at a position much nearer to the node of the Ω_+ mode than of the Ω_- mode. The Ω_+ mode will therefore be less strongly excited than the Ω_- mode. To obtain the resonances at approximately equal amplitudes for Fig. 7, we purposely compensated for this effect by raising the tuning of the uncoupled string resonance to come closer to the Ω_+ resonance.

To consider how the resonances of the coupled systems vary with tuning of the strings, we have plotted some typical dispersion curves, Fig. 8a and 8b, for strong and weak coupling situations respectively. These curves again underline the very different behaviour of the system in the strongly and weakly coupled regimes. In these curves, the solid lines represent the frequencies of the normal modes of the coupled system and the shaded areas the width of their resonances at half-height.

The measurements shown in Fig. 7 were obtained at a position represented schematically by the line A'B' in Fig. 8a. As the tuning of the string moves away from f_B, there is an initially small, second-order increase in the separation of the normal modes and the damping of the two modes becomes unequal, though the sum of their widths remains unchanged. For excitation near one of the ends, the zero-crossing frequencies will depend both on the position of excitation and on the mistuning; in general, the zero-crossing frequencies will not be symmetrically placed about f_B, or even about the average of the normal mode frequencies excited, as was assumed in the simplest Schelleng model calculation[1].

The dispersion curves shown in Figs. 8a and 8b also demonstrate why relevant information about the important resonances of an instrument can be obtained even when the strings are not tuned to their normal playing pitches. Apart from a small, easily calculated shift in resonant frequencies, which depends on the frequen-

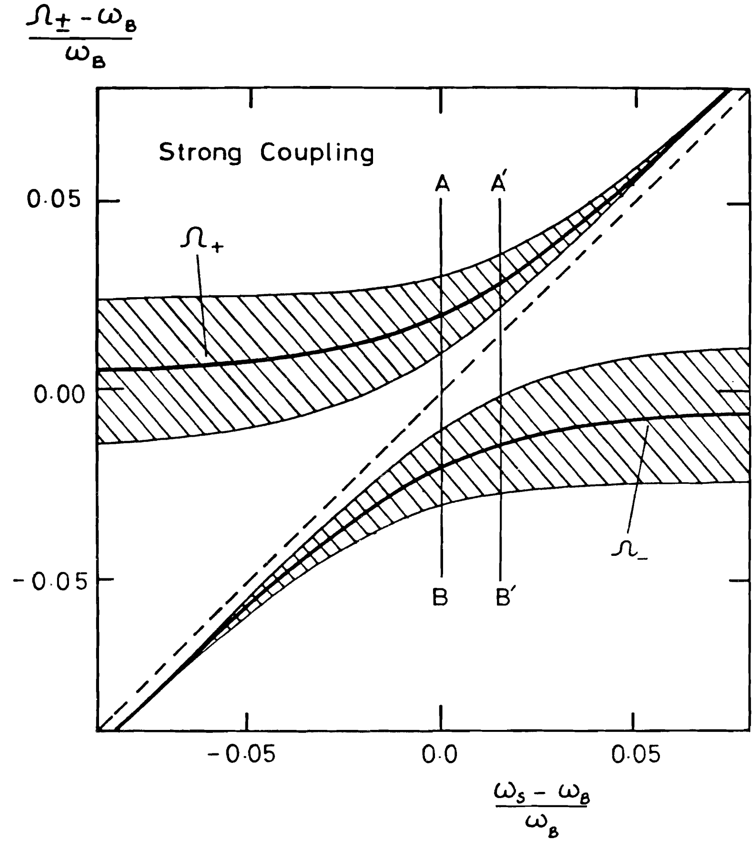

FIG.8(a): Typical dispersion curves for normal modes in the strong coupling limit

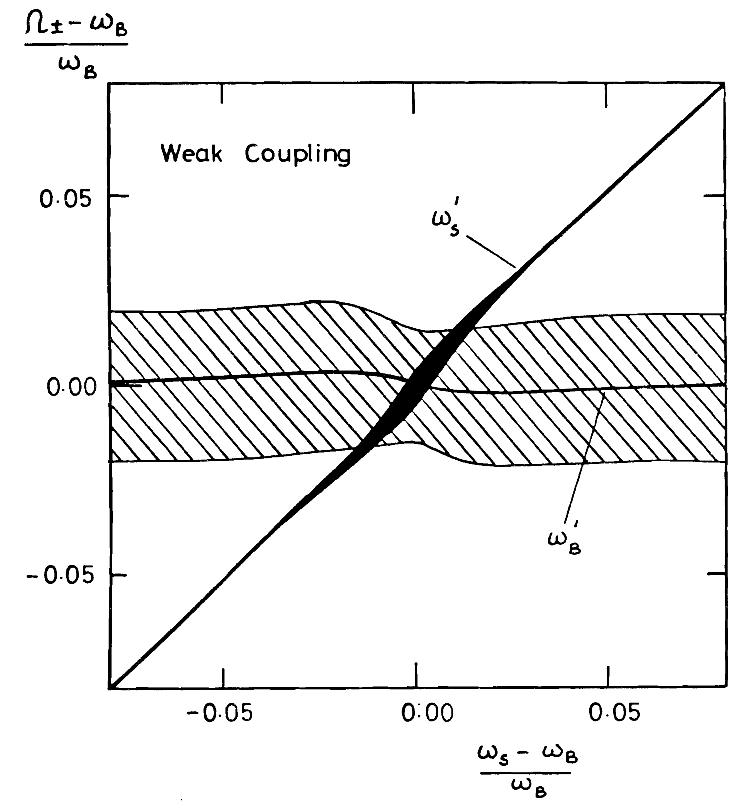

FIG.8(b): Typical dispersion curves for normal modes in low coupling limit

cies of the string resonances relative to the structural resonances, the tuning of the strings has very little affect on the structural properties, unless the frequency of a particular string resonance lies within about a half-width of a structural resonance, where the effects of the interaction are then described by the theory outlined above.

Returning to the problem of the bowed string, we note that when a string is bowed the string is forced to move in the direction of the bowing action at the point of string excitation. No direct comparison can therefore be made between any theory for the bowed-string and our measurements of admittance curves, since this additional boundary condition is not satisfied in the experiments described so far. We note that, without such a constraint, the additional complexity introduced by both "coupled" and "uncoupled" modes of string vibration can cause the imaginary component of the admittance to pass through zero five rather than three times, as is seen in Fig. 7.

When a bow is laid across the string and measurements are made on the length of string resonating between bow and nut, we observe a very complicated resonant response, with a number of non-linear and somewhat irreproducible features, as illustrated by the measurements shown in Fig. 9. The resonances marked A can be identified with the resonances of the length of string between the nut and the bow when "sticking" takes place; the resonances marked B are almost certainly associated with the resonating length of string between nut and bridge "slipping", beneath the bow and therefore being heavily damped; the resonances marked C, D and E are associated with resonances of the bow hair and are rather similar to the resonances reported in a study of the characteristic vibrations of the bow by Schumacher[15]. We have not attempted to analyze these measurements in any detail and simply include them to suggest that measurements of string resonances might provide a useful method for studying certain aspects of the bow-string interaction.

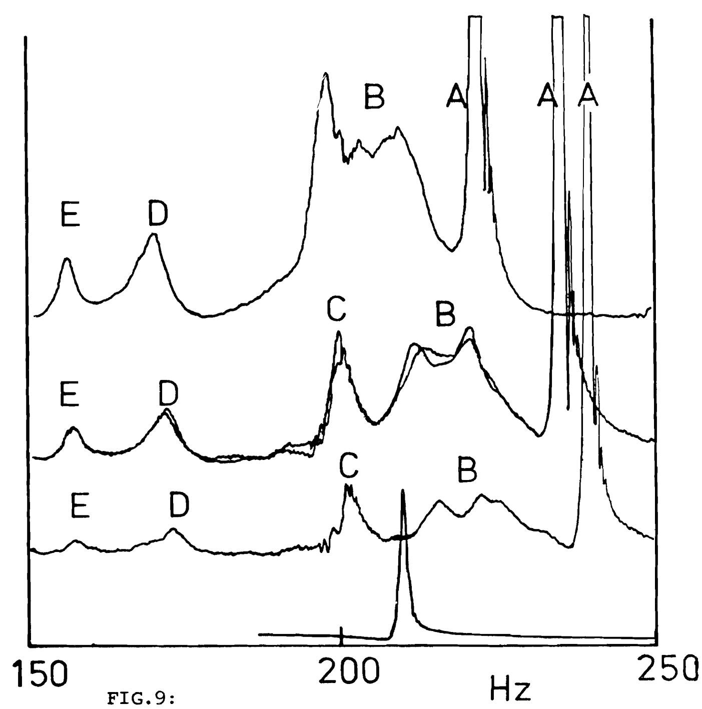

FIG.9:
"String resonances" with bow in normal bowing position for three different measuring currents, the upper trace corresponding to the largest current illustrating the effects of heating. The lowest trace at reduced sensitivity was obtained for the largest measuring current but with the bow removed.

High sensitivity measurements.

As a final example of the information that can be obtained from string resonances, in Fig. 10, we show two sets of traces obtained at high sensitivity for the n=2 resonance of the C-string on the cello used in obtaining the measurements shown in Fig. 7. The string was again excited in a direction close to the principal coupling direction, so that the curves are dominated by the "coupled" string resonances. B is the weakly excited "uncoupled" string resonance and A and C are the split resonances of the coupled system, now excited at different amplitudes. In addition there is a further weak resonance D at a lower frequency. This much narrower resonance was also subsequently found to give a pronounced wolfnote, but over such a narrow range of frequencies that it would rarely prove a problem in practice. The upper trace was obtained with a small additional mass of a few grams placed on the belly, at the foot of the bridge nearest the C-string. Comparison of the two sets of measurements shows that the upper resonance is very much more strongly affected by the additional loading than the lower resonance. Such observations are similar to Firth's measurements[16] made using a rather different technique.

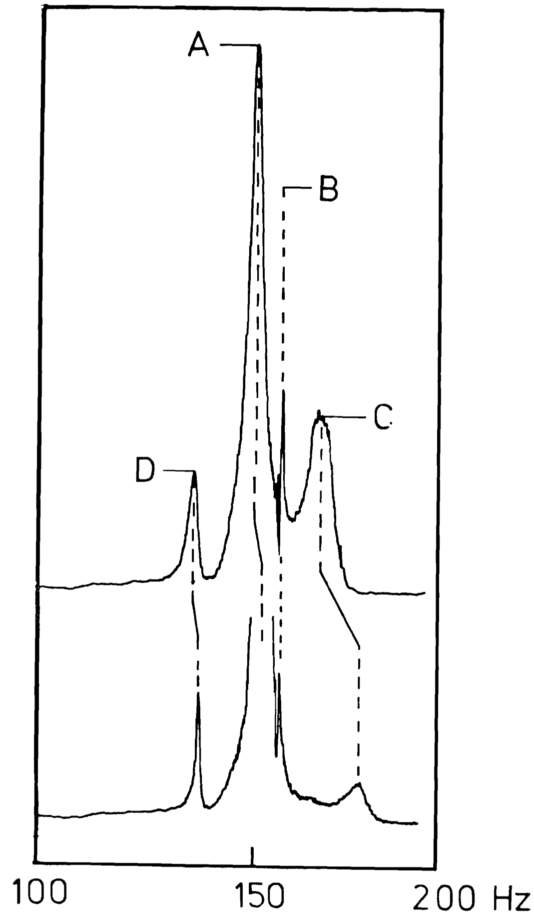

FIG.10: n=2 C-string resonances on a cello, midway between two strongly coupled structural resonances. An additional mass has been placed on the belly for the upper set of measurements.

Summary.

We hope that this brief account of our measurements will have given some idea of the potential of this technique for obtaining reliable quantitative information about many of the structural properties of an instrument that ultimately determine its tone. We hope to use this technique to make a systematic study of a number of violins and cellos, to investigate any obvious correlation with their musical quality. We also hope to study the way that the structural properties are affected by such factors as the humidity and, by comparing measurements of Q_B on instruments under normal conditions and in an evacuated chamber, to determine the radiation efficiency of particular structural resonances.

Acknowledgements.

I should like to acknowledge the help provided over several years from undergraduate students, whose project work[17] laid the foundation for the measurements we present here. I am also particularly grateful for the advice and encouragement received from Maurice Hancock, Michael McIntyre, Arthur Benade, Gabriel Weinreich and Carleen Hutchins. I should also like to thank Robert Schumacher for his helpful comments on an earlier version of this paper. This work is supported by a grant from the Royal Society.

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